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Two-state representations of three-state neural networks

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Abstract. We investigate to what extent the dynamical behaviour of nets consisting of three-state neurons can be realised in nets using two-state neurons. We find that the generalisation from a two-state neuron to a three-state neuron cannot introduce any new behaviour in the deterministic case, but can do so for noisy nets, even in the zero-noise limit. For those noisy three-state nets which have a Hamiltonian description and a two-state representation we investigate the relationship between the three-state Hamiltonian and the two-state Hamiltonian, and find an interesting level-splitting phenomenon.

1. Introduction

One of the most widely studied, and best understood, neural network models is the binary threshold model of Hopfield and Little (Hopfield 1982, Little 1974). Recently there has been growing interest in generalising such networks to ones with three-state neurons (e.g. Meunier *et al* 1989 (MHV) and Yedidia 1989). In particular MHV claim that such three-state nets offer superior performance for a variety of problems.

The aim of this paper is to investigate to what extent such three-state networks exhibit behaviour fundamentally different from two-state ones. In particular we examine when the dynamical behaviour of a three-state net can be duplicated by some appropriately chosen two-state net. *A priori*, we would expect that the generalisation from two-state neurons to three-state neurons ought to lead to a wider class of possible behaviour. It turns out that for deterministic nets this is not so, and it is very easy to see that, for instance, every three-state net of the type studied by MHV and by Yedidia can be rewritten as a two-state net with twice the number of neurons.

On the other hand, when noise is added we find that, in general, given a three-state net there is no two-state net with equivalent (probabilistic) dynamics. This can be true for even arbitrarily low noise levels; thus there are three-state nets whose deterministic behaviour can be represented by a two-state net, but as soon as an arbitrarily small amount of noise is added, no such representation is possible. This is, for instance, the case for the MHV net if the activations are sufficiently high. The Yedidia net, on the other hand, does always have a two-state representation for sufficiently low noise levels.

Lastly, for those three-state nets (such as the Yedidia model) which have a Hamiltonian description (Peretto 1984), we examine the relationship between the three-state Hamiltonian and the two-state Hamiltonian. We find that in general they are not equal, and the two-state Hamiltonian exhibits an interesting phenomenon of level splitting with respect to the three-state Hamiltonian. This is due to the fact that the two-state representation is not one-to-one; that is, each state of the three-state net corresponds to more than one state of the equivalent two-state net. In general the energies of all these latter states will be different. As one might expect, however, the two Hamiltonians do have the same ground states.

2. Deterministic behaviour

We shall consider networks of N neurons with connection weights $\omega_{ij} \in \mathbb{R}$. The state of neuron i at time t is $a_i(t)$. We assume that the net evolves in discrete time steps, i.e. that t is an integer. Throughout this paper we shall assume parallel dynamics; that is, all the a_i are updated simultaneously. By far the most common two-state deterministic dynamics studied is a simple threshold model:

$$a_i(t+1) = \begin{cases} 1 & \text{if } h_i(t) \geqslant V_i \\ -1 & \text{if } h_i(t) < V_i \end{cases} \tag{2.1}$$

where

$$h_i(t) = \sum_{j=1}^{N} \omega_{ij} a_j(t)$$

with $a_i(t) \in \{-1, 1\}$, and V_i is a fixed threshold. The natural generalisation to a three-state net is to consider $a_i(t) \in \{-1, 0, 1\}$, define a second threshold $U_i \leqslant V_i$ (though often one takes $U_i = -V_i$, with $V_i \geqslant 0$), and set

$$a_i(t+1) = \begin{cases} 1 & \text{if } h_i(t) \geqslant V_i \\ 0 & \text{if } U_i \leqslant h_i(t) < V_i \\ -1 & \text{if } h_i(t) < U_i \end{cases} \tag{2.2}$$

with h_i as before. Given such a three-state net, it is a trivial observation that there is a two-state net with exactly the same dynamics. Namely, we replace each three-state neuron (in state a_i) by a pair of two-state neurons in states s_i and $\tilde{s}_i$, so that $a_i = \frac{1}{2}(s_i + \tilde{s}_i)$. Thus we take

$$\begin{aligned} s_i(t+1) &= \begin{cases} 1 & \text{if } h_i(t) \geqslant V_i \\ -1 & \text{if } h_i(t) < V_i \end{cases} \\ \tilde{s}_i(t+1) &= \begin{cases} 1 & \text{if } h_i(t) \geqslant U_i \\ -1 & \text{if } h_i(t) < U_i \end{cases} \end{aligned} \tag{2.3}$$

with

$$h_i(t) = \sum_{j=1}^{N} \tfrac{1}{2}\omega_{ij}(s_j(t) + \tilde{s}_j(t)).$$

Clearly this is still a simple binary threshold net with thresholds V_i and U_i and connection weights $\frac{1}{2}\omega_{ij}$ between either of s_i and $\tilde{s}_i$ to either of s_j and $\tilde{s}_j$. Note that since we always have $s_i \leqslant \tilde{s}_i$, the state $(1, -1)$ of a pair is impossible. We thus have a one-to-one correspondence between states of the three-state net and the two-state representation. Thus the dynamics of a_i will be exactly the same as the dynamics of $\frac{1}{2}(s_i + \tilde{s}_i)$.

Also observe that any two-state net can be realised as a three-state net simply by setting $V_i = U_i$. Thus the class of possible behaviours for such three-state networks is exactly the same as the class of possible behaviours for two-state networks. If, however, we require $V_i = -U_i$ (which MHV and Yedidia do) then this is only possible for $V_i = 0$, in which case the three-state nets form a proper subclass of all possible two-state nets.

One natural generalisation of the two-state representation is to allow different inputs to s_i and $\tilde{s}_i$. Thus we would replace the equation for $\tilde{s}_i$ by

$$\tilde{s}_i(t+1) = \begin{cases} 1 & \text{if } \tilde{h}_i(t) \geq U_i \\ -1 & \text{if } \tilde{h}_i(t) < U_i \end{cases}$$

with

$$\tilde{h}_i(t) = \sum_{j=1}^{N} \tfrac{1}{2}\tilde{\omega}_{ij}(s_j(t) + \tilde{s}_j(t)).$$

Note that in general we no longer have $s_i \leq \tilde{s}_i$, and hence the state $(1, -1)$ can now be attained. However, the dynamics preserves the essential features of three-state behaviour since the evolution depends only on $a_i(t) = (s_i(t) + \tilde{s}_i(t))$ which can only be in one of three states. The state $a_i(t) = 0$ which corresponds to a lack of information thus now has two representations $(-1, 1)$ and $(1, -1)$. This generalisation might be useful since it allows the rules for the $a_i(t) = 1$ state and the $a_i(t) = -1$ state to be asymmetrical, but note that it cannot be written in a simple manner as a net with three-state neurons.

We can obviously use the same technique for other types of deterministic three-state behaviour, though the resulting two-state coding will not always be so elegant.

3. Noisy three-state nets

In the most general case a noisy net is determined by the Markov chain which specifies the transition probabilities amongst all the possible states of the net. Denote the state of the whole net by $\boldsymbol{a}(t) = (a_1(t), \ldots, a_N(t))$, where $a_i(t)$ is the state of the ith neuron at time t. Then for a three-state net of N neurons the Markov chain is given by the $3^N \times 3^N$ matrix with entries $\mathbf{P}_{bc} = \mathcal{P}(\boldsymbol{a}(t+1) = \boldsymbol{c} | \boldsymbol{a}(t) = \boldsymbol{b})$ defining the probability that the net is in state $\boldsymbol{c}$ at time $t+1$ given that it was in state $\boldsymbol{b}$ at time t. In general not every Markov chain $\mathbf{P}_{bc}$ gives a neural net; this is because for a realistic net we expect the conditional firing probabilities of distinct neurons to be independent (as random variables). In other words, we require

$$\mathcal{P}(a_i(t+1) = c_i \wedge a_j(t+1) = c_j | \boldsymbol{a}(t) = \boldsymbol{b})$$
$$= \mathcal{P}(a_i(t+1) = c_i | \boldsymbol{a}(t) = \boldsymbol{b}) \mathcal{P}(a_j(t+1) = c_j | \boldsymbol{a}(t) = \boldsymbol{b})$$

with

$$\mathcal{P}(a_i(t+1) = c_i \wedge a_j(t+1) = c_j | \boldsymbol{a}(t) = \boldsymbol{b}) = \sum_{\boldsymbol{d}: d_i = c_i, d_j = c_j} \mathbf{P}_{bd}$$

$$\mathcal{P}(a_i(t+1) = c_i | \boldsymbol{a}(t) = \boldsymbol{b}) = \sum_{\boldsymbol{d}: d_i = c_i} \mathbf{P}_{bd}$$

and similarly for $\mathcal{P}(a_j(t+1) = c_j | \boldsymbol{a}(t) = \boldsymbol{b})$. Note that this implies that

$$\mathbf{P}_{bc} = \prod_{i=1}^{N} \mathcal{P}(a_i(t+1) = c_i | \boldsymbol{a}(t) = \boldsymbol{b}).$$

Given such a noisy three-state net, we again attempt to find an equivalent two-state net by replacing each three-state neuron by two two-state ones. It remains to define the two-state net transition probabilities. At first it might seem that we can do this by

$$\mathcal{P}(\boldsymbol{S}(t+1)=\boldsymbol{X}|\boldsymbol{S}(t)=\boldsymbol{Y})=\mathbf{P}_{\psi(\boldsymbol{Y})\psi(\boldsymbol{X})}$$

where $\boldsymbol{S}=(S_1,\ldots,S_N)$ is the state of the two-state net, with $S_i=(s_i,\tilde{s}_i)$, and ψ is a function from the states of the two-state net to the states of the three-state net, given by

$$\psi(\boldsymbol{S})=(\tfrac{1}{2}(s_1+\tilde{s}_1),\ldots,\tfrac{1}{2}(s_N+\tilde{s}_N)).$$

There are, however, two problems with this. The first is that (unlike in the deterministic case) ψ is in general not one-to-one, and thus

$$\sum_{\boldsymbol{X}}\mathbf{P}_{\psi(\boldsymbol{Y})\psi(\boldsymbol{X})}>1.$$

We can overcome this by spreading $\mathbf{P}_{\psi(\boldsymbol{Y})\psi(\boldsymbol{X})}$ amongst all the states $\boldsymbol{X}'$ with $\psi(\boldsymbol{X}')=\psi(\boldsymbol{X})$. There will be $2^{m(\boldsymbol{X})}$ such states, where $m(\boldsymbol{X})=\#\{i\colon x_i=-\tilde{x}_i\}$ and hence we have a large choice of how to do this. This brings up the second problem: do any of these choices give a two-state net with independent neuron firing probabilities. Clearly we can choose s_i and $\tilde{s}_i$ to be independent of s_j and $\tilde{s}_j$ for $i\neq j$. The question is thus whether or not s_i can be chosen independent of $\tilde{s}_i$. It turns out that in general this is not so. To see this, let us fix a particular state $\boldsymbol{a}(t)=\boldsymbol{b}$ for the three-state net and $\boldsymbol{S}(t)=\boldsymbol{Y}$ with $\psi(\boldsymbol{Y})=\boldsymbol{b}$, and consider the ith neuron. Let

$$\mathcal{P}(a_i(t+1)=+1)=\alpha \qquad \mathcal{P}(a_i(t+1)=-1)=\beta \qquad \mathcal{P}(a_i(t+1)=0)=\gamma$$

and

$$\mathcal{P}(s_i(t+1)=1)=p \qquad \mathcal{P}(\tilde{s}_i(t+1)=1)=q$$

where all probabilities are conditional on $\boldsymbol{b}$ and $\boldsymbol{Y}$ respectively. Then if s_i and $\tilde{s}_i$ are to be independent, p and q must satisfy the equations

$$pq=\alpha \qquad (1-p)(1-q)=\beta \qquad (1-p)q+(1-q)p=\gamma. \tag{3.1}$$

This occurs if and only if p and q are the roots of the quadratic

$$f(\lambda)=\lambda^2+(\beta-\alpha-1)\lambda+\alpha$$

which gives

$$p,q=\tfrac{1}{2}(\gamma+2\alpha\pm\sqrt{\gamma^2-4\alpha\beta}). \tag{3.2}$$

We thus see that if $\gamma^2<4\alpha\beta$, there are no real solutions of equation (3.1). On the other hand, if $\gamma^2\geqslant4\alpha\beta$, it is easy to check that both roots of f lie in the interval $[0,1]$, and hence give valid firing probabilities for s_i and $\tilde{s}_i$. This is because $f(0)=\alpha>0$, $f(1)=\beta>0$, the minimum of f is taken at $\lambda_0=\tfrac{1}{2}(1+\alpha-\beta)\in[0,1]$. Thus we see that the three-state neuron can be represented by two independent two-state neurons if and only if

$$\gamma^2\geqslant4\alpha\beta. \tag{3.3}$$

The dividing line $\gamma^2=4\alpha\beta$ can be written in terms of α and β as $(\alpha-\beta)^2=2(\alpha+\beta-\tfrac{1}{2})$; this is a parabola passing through $(1,0)$, $(\tfrac{1}{4},\tfrac{1}{4})$ and $(0,1)$.

The conditional probabilities α, β and γ will of course depend on the state $\boldsymbol{b}$ and the neuron i. For an arbitrary three-state net we cannot expect (3.3) to be satisfied for every such choice of state and neuron. We thus conclude that, in contrast to the deterministic case, noisy three-state nets cannot in general be represented by two-state nets.

4. Low-noise case

Given the above discrepancy between deterministic nets and noisy ones, we might at least hope that if the noise in the three-state net is sufficiently small (so that we are close to the deterministic case) we can always find an equivalent two-state representation. It turns out that even this is not the case. We shall consider two slightly different noisy three-state models, namely those defined by MHV and by Yedidia (1989). As before, let

$$h_i(t) = \sum_{j=1}^{N} \omega_{ij} a_j(t)$$

for some fixed set of weights ω_{ij} and thresholds $V_i > 0$. Then the MHV model assigns the following conditional firing probabilities to the ith neuron:

$$\begin{aligned}
\mathscr{P}(a_i(t+1) = +1) &= \frac{1}{1+\exp[-(h_i(t) - V_i)/\tau]} \\
\mathscr{P}(a_i(t+1) = -1) &= \frac{1}{1+\exp[(h_i(t) + V_i)/\tau]} \\
\mathscr{P}(a_i(t+1) = 0) &= 1 - \frac{1}{1+\exp[-(h_i(t) - V_i)/\tau]} - \frac{1}{1+\exp[(h_i(t) + V_i)/\tau]}
\end{aligned} \tag{4.1}$$

whilst the Yedidia net has:

$$\begin{aligned}
\mathscr{P}(a_i(t+1) = +1) &= \frac{\exp[(h_i(t) - V_i)/\tau]}{z_i} \\
\mathscr{P}(a_i(t+1) = -1) &= \frac{\exp[-(h_i(t) + V_i)/\tau]}{z_i} \\
\mathscr{P}(a_i(t+1) = 0) &= \frac{1}{z_i}
\end{aligned} \tag{4.2}$$

where

$$z_i = 1 + \exp[(h_i(t) - V_i)/\tau] + \exp[-(h_i(t) + V_i)/\tau].$$

Here, τ is a noise or 'temperature' parameter and as $\tau \to 0$ the noise in the net tends to zero. Note that in the limit $\tau \to 0$ both of the above models give the deterministic dynamics described in section 2 (with $U_i = -V_i$). For small τ we can thus regard them as noisy versions of the deterministic net.

Let us first consider the MHV model. Define α, β and γ as in section 3, and let

$$\Delta_i(h_i) = \gamma^2 - 4\alpha\beta.$$

A straightforward calculation yields

$$\Delta_i(h_i) = \frac{4\sinh^2(V_i/\tau) - 8\exp(-V_i/\tau)[\cosh(V_i/\tau) + \cosh(h_i/\tau)]}{4[\cosh(V_i/\tau) + \cosh(h_i/\tau)]^2}.$$

Now, the dominant terms in the numerator as $\tau \to 0$ are, respectively,

$$\begin{aligned}
&\exp(2V_i/\tau) && \text{if } |h_i| < 3V_i \\
&-4\exp[(h_i - V_i)/\tau] && \text{if } h_i \geqslant 3V_i \\
&-4\exp[-(h_i + V_i)/\tau] && \text{if } h_i \leqslant -3V_i.
\end{aligned}$$

In order to satisfy (3.3), we want $\Delta_i(h_i) \geqslant 0$ for all sufficiently small τ, and hence require the leading term to be $\exp(2V_i/\tau)$. Thus a given MHV net has a two-state representation for sufficiently low noise if and only if the activation h_i is reasonably small, i.e. $|h_i| < 3V_i$, for all neurons i and all possible states $\boldsymbol{a}(t)$. We therefore conclude that even in the case of low noise many MHV three-state nets are not representable by two-state models.

On the other hand, for Yedidia's model we have

$$\Delta_i(h_i) = \frac{1 - 4\exp(-2V_i/\tau)}{z_i^2}.$$

Clearly for τ sufficiently small we have $\Delta_i \geqslant 0$, and in fact we can choose τ uniformly in i and h_i. We then get the following two-state firing probabilities:

$$\begin{aligned}\mathscr{P}(s_i(t+1)=1) &= \frac{1+2\exp[(h_i - V_i)/\tau] - \sqrt{1-4\exp(-2V_i/\tau)}}{2z_i} \\ \mathscr{P}(\tilde{s}_i(t+1)=1) &= \frac{1+2\exp[(h_i - V_i)/\tau] + \sqrt{1-4\exp(-2V_i/\tau)}}{2z_i}\end{aligned} \tag{4.3}$$

with z_i defined as before. In the limit $\tau \to 0$, this gives the deterministic two-state dynamics described in section 2, as expected. Also note that as $\tau \to 0$, this model approaches the well known model of Little (1974) with thresholds V_i and $-V_i$ for s_i and $\tilde{s}_i$ respectively:

$$\begin{aligned}\mathscr{P}(s_i(t+1)=1) &= \frac{1}{1+\exp[-(h_i(t) - V_i)/\tau]} \\ \mathscr{P}(\tilde{s}_i(t+1)=1) &= \frac{1}{1+\exp[-(h_i(t) + V_i)/\tau]}.\end{aligned} \tag{4.4}$$

This is, of course, a noisy version of the dynamics given by (2.3) with $U_i = -V_i$. Observe that unlike the Little model, the distributions (4.3) are not symmetric about the thresholds $-V_i$ and V_i.

One way to ensure that there is a two-state representation for all values of τ is to reverse the above procedure, and construct a three-state net from a two-state one. Thus, starting with (4.4) we get the three-state model with probabilities

$$\begin{aligned}\mathscr{P}(a_i(t+1)=+1) &= \frac{\exp(h_i(t)/\tau)}{z_i} \\ \mathscr{P}(a_i(t+1)=-1) &= \frac{\exp(-h_i(t)/\tau)}{z_i} \\ \mathscr{P}(a_i(t+1)=0) &= \frac{\exp(-V_i/\tau)+\exp(V_i/\tau)}{z_i} = \frac{2\cosh(V_i/\tau)}{z_i}\end{aligned} \tag{4.5}$$

with

$$\begin{aligned}z_i &= \exp(V_i/\tau) + \exp(-V_i/\tau) + \exp(h_i(t)/\tau) + \exp(-h_i(t)/\tau) \\ &= 2\cosh(V_i/\tau) + 2\cosh(h_i(t)/\tau)\end{aligned}$$

with $h_i(t)$ defined as usual. We shall call this the Little three-state model. Given that the standard two-state Little model is so well understood, and there is a wide variety of techniques available to study it, (4.5) would seem to be a good choice for the study of noisy three-state nets. It would be interesting to see whether its behaviour differed in any marked respect from the MHV (4.1) and Yedidia (4.2) models. Certainly all three have the same zero-noise limit; however, this limit is approached in different ways. This, incidentally, accounts for the fact that (4.2) and (4.5) can be realised as two-state nets, whilst (4.1) cannot.

Finally, note that, unlike in the deterministic case, when a two-state representation is possible, it is not one-to-one. Each state $\boldsymbol{b}$ of the three-state net corresponds to $2^{m(\boldsymbol{b})}$ states of the two-state representation, where $m(\boldsymbol{b}) = \#\{i: a_i = 0\}$. These are precisely the states $\boldsymbol{X}$ such that $\psi(\boldsymbol{X}) = \boldsymbol{b}$. The transition probability $\mathbf{P}_{XY}$ to some other state $\boldsymbol{Y}$ is exactly the same for all these states, i.e. $\mathbf{P}_{XY} = \mathbf{P}_{X'Y}$ for all $\boldsymbol{X}'$ such that $\psi(\boldsymbol{X}) = \psi(\boldsymbol{X}')$, and

$$\mathbf{P}_{bc} = \sum_{Y:\psi(Y)=c} \mathbf{P}_{XY}$$

for any $\boldsymbol{X}$ such that $\psi(\boldsymbol{X}) = \boldsymbol{b}$.

5. Hamiltonian formulation

A particularly useful tool in the study of noisy neural nets has been the energy or Hamiltonian description of Peretto (1984). This can be applied to any net for which the associated Markov chain is irreducible, aperiodic (e.g. Feller 1968) and satisfies detailed balance, i.e. for which there exists some function F of the states such that

$$\frac{\mathbf{P}_{bc}}{\mathbf{P}_{cb}} = \frac{F(\boldsymbol{c})}{F(\boldsymbol{b})}.$$

For such a net, let $\rho(\boldsymbol{b}, t)$ be the probability that the net is in state $\boldsymbol{b}$ at time t (given some fixed initial condition $\boldsymbol{a}(0)$ at $t = 0$). Then a standard result on Markov chains states that, irrespective of the initial state $\boldsymbol{a}(0)$, we have

$$\lim_{t\to\infty} \rho(\boldsymbol{b}, t) = \frac{F(\boldsymbol{b})}{\Sigma_c F(\boldsymbol{c})}$$

where the sum is over all possible states $\boldsymbol{c}$. Writing $H_\tau(\boldsymbol{b}) = -\tau \log F(\boldsymbol{b})$, we see that this limiting distribution is a Boltzmann one, with effective 'Hamiltonian' $H_\tau(\boldsymbol{b})$ given by

$$\lim_{t\to\infty} \rho(\boldsymbol{b}, t) = \frac{\exp(-H_\tau(\boldsymbol{b})/\tau)}{\Sigma_c \exp(-H_\tau(\boldsymbol{c})/\tau)}.$$

Such a formalism allows one to apply many of the standard techniques of statistical mechanics to the analysis of neural nets. In particular, in the limit $\tau \to 0$ the net will settle into one of the ground states of $\lim_{\tau\to 0} H_\tau(\boldsymbol{b})$. Thus the minima of $\lim_{\tau\to 0} H_\tau(\boldsymbol{b})$ determine the low-noise, long-time dynamics of the net.

In this section we ask what the relationship is between the Hamiltonian $H^3_\tau(\boldsymbol{b})$ of a three-state net and the Hamiltonian $H^2_\tau(\boldsymbol{X})$ of its two-state representation, assuming that they both exist. Naively one might expect that $H^2_\tau(\boldsymbol{X}) = H^3_\tau(\psi(\boldsymbol{X}))$ for all states $\boldsymbol{X}$. However, recall that ψ is in general not one-to-one, and the correct relationship between the two Hamiltonians must thus be

$$\rho^{\infty}(\boldsymbol{b}) = \sum_{\boldsymbol{X}:\psi(\boldsymbol{X})=\boldsymbol{b}} \rho^{\infty}(\boldsymbol{X}) \tag{5.1}$$

where

$$\rho^{\infty}(\boldsymbol{b}) = \lim_{t\to\infty} \rho(\boldsymbol{b}, t) = \frac{\exp(-H_\tau^3(\boldsymbol{b})/\tau)}{\Sigma_{\boldsymbol{c}} \exp(-H_\tau^3(\boldsymbol{c})/\tau)}$$

$$\rho^{\infty}(\boldsymbol{X}) = \lim_{t\to\infty} \rho(\boldsymbol{X}, t) = \frac{\exp(-H_\tau^2(\boldsymbol{X})/\tau)}{\Sigma_{\boldsymbol{Y}} \exp(-H_\tau^2(\boldsymbol{Y})/\tau)}.$$

In general, there is no reason why any two states $\boldsymbol{X}$ and $\boldsymbol{X}'$ with $\psi(\boldsymbol{X}') = \psi(\boldsymbol{X})$ should have the same energy. Indeed, for small τ we expect a wide variation in the energy of the states in $\{\boldsymbol{X}'$: $\psi(\boldsymbol{X}') = \psi(\boldsymbol{X})\}$. This is because in the deterministic limit only the state $\boldsymbol{X}_0$ with $x_i \leqslant \tilde{x}_i$, for all i, is allowed. Thus only this state should enter into the sum in (5.1), and all the other states should have much higher energy. As we shall see below, this 'level splitting' is precisely the behaviour that we observe in the Yedidia and Little models, and in fact for these nets $\boldsymbol{X}_0$ turns out to have minimum energy amongst the states in $\{\boldsymbol{X}'$: $\psi(\boldsymbol{X}') = \psi(\boldsymbol{X})\}$ for all values of τ for which there is a two-state representation. Furthermore we find that

$$\lim_{\tau\to 0} H_\tau^3(\psi(\boldsymbol{X}_0)) = \lim_{\tau\to 0} H_\tau^3(\boldsymbol{X}_0).$$

This is a stronger condition than H_τ^3 and H_τ^2 just having the same ground states.

Note, however, that we have not been able to prove these results for an arbitrary three-state net with Hamiltonian and two-state representation, nor have we been able to show that if a three-state net has a Hamiltonian, then so does its two-state representation (when it exists) or vice versa.

First let us see which of the noisy nets considered above have Hamiltonian descriptions. From now on we shall only consider symmetric connections, i.e. $\omega_{ij} = \omega_{ji}$ for all i, j. There seems little hope of detailed balance being satisfied for asymmetric nets in general. Also observe that all the noisy nets considered in this paper have $\mathbf{P}_{\boldsymbol{bc}} > 0$ for all states $\boldsymbol{b}$, $\boldsymbol{c}$, and hence are aperiodic and irreducible. It thus remains to verify detailed balance. In what follows we shall write

$$h_i(\boldsymbol{b}) = \sum_{j=1}^{N} \omega_{ij} b_j$$

for three-state nets and

$$h_i(\boldsymbol{X}) = \sum_{j=1}^{N} \tfrac{1}{2}\omega_{ij}(x_i + \tilde{x}_i)$$

for two-state nets.

MHV were unable to find a Hamiltonian for their model, and in fact it is not difficult to show that their nets do not in general satisfy detailed balance except for trivial choices of weights. All the other nets considered in sections 3 and 4 turn out to have Hamiltonians. First recall that the standard Little model (of N neurons with thresholds V_i) has the Hamiltonian (Peretto 1984)

$$H_\tau^{\mathrm{L}}(\boldsymbol{S}) = \sum_{i=1}^{N} \tfrac{1}{2} s_i V_i - \tau \sum_{i=1}^{N} \log\cosh\left[\left(\sum_{j=1}^{N} \omega_{ij} s_j - V_i\right) \frac{1}{2\tau}\right].$$

Applying this to (4.4) with $2N$ neurons with thresholds V_i and $-V_i$, we get

$$H_\tau^{\mathrm{L},2}(\boldsymbol{X}) = \sum_{i=1}^{N} \tfrac{1}{2}(x_i - \tilde{x}_i) V_i - \tau \sum_{i=1}^{N} \log\cosh[(h_i(\boldsymbol{X}) - V_i)/2\tau]$$

$$- \tau \sum_{i=1}^{N} \log\cosh[(h_i(\boldsymbol{X}) + V_i)/2\tau]$$

$$= \sum_{i=1}^{N} \tfrac{1}{2}(x_i - \tilde{x}_i) V_i - \tau \sum_{i=1}^{N} \log[\cosh(h_i(X)/\tau) + \cosh(V_i/\tau)]$$

where we have ignored the constant term $\tau N \log 2$. Observe that this cannot be written in terms of $b_i = \frac{1}{2}(x_i + \tilde{x}_i)$, and hence does not directly give a Hamiltonian for the three-state Little model (4.5). However, working directly from (4.5) we get

$$H_\tau^{\mathrm{L},3}(\boldsymbol{b}) = -\tau \sum_{i: b_i = 0} \log\cosh(V_i/\tau) - \tau \sum_{i=1}^{N} \log[\cosh(h_i(\boldsymbol{b})/\tau) + \cosh(V_i/\tau)].$$

Note that $H_\tau^{\mathrm{L},2}(\boldsymbol{X})$ and $H_\tau^{\mathrm{L},3}(\boldsymbol{b})$ exhibit precisely the level splitting described above. Now let us consider the Yedidia model. We can rewrite (4.2) as

$$\mathscr{P}(a_i(t+1) = c_i) = \frac{\exp(c_i h_i(t)/\tau) \exp(-|c_i| V_i/\tau)}{z_i}$$

for $c_i \in \{-1, 0, 1\}$. A straightforward calculation then yields

$$H_\tau^{\mathrm{Y},3}(\boldsymbol{b}) = \sum_{i=1}^{N} |b_i| V_i - \tau \sum_{i=1}^{N} \log z_i(\boldsymbol{b})$$

for the three-state Yedidia model, with $z_i(\boldsymbol{b})$ as in (4.2). On the other hand, the probability distributions of the two-state representation (4.3) look quite different to those known to allow Hamiltonians, and hence we initially did not expect one to exist for this model. Remarkably, if we pair up the neurons x_i and $\tilde{x}_i$ and set $X_i = (x_i, \tilde{x}_i)$ as before, we get

$$\mathscr{P}(S_i(t+1) = (+1, +1)) = \frac{\exp[(h_i(t) - V_i)/\tau]}{z_i}$$

$$\mathscr{P}(S_i(t+1) = (-1, -1)) = \frac{\exp[-(h_i(t) + V_i)/\tau]}{z_i}$$

$$\mathscr{P}(S_i(t+1) = (-1, +1)) = \frac{1 + R_i}{2 z_i}$$

$$\mathscr{P}(S_i(t+1) = (+1, -1)) = \frac{1 - R_i}{2 z_i}$$

where

$$R_i = \sqrt{1 - 4\,\mathrm{e}^{-2V_i/\tau}}$$

$$z_i = 1 + \exp[(h_i(t) - V_i)/\tau] + \exp[-(h_i(t) + V_i)/\tau]$$

and as before $h_i(t) = h_i(\boldsymbol{S}(t))$. Observe that the $X_i = (1, -1)$ and $X_i = (-1, 1)$ cases only depend on $h_i(t)$ through the denominator z_i, which is identical for all four states. This fortuitous cancellation allows us to write

$$\mathscr{P}(S_i(t+1) = X_i) = \frac{\exp(c_i h_i(t)/\tau) \exp(-|c_i| V_i/\tau)}{z_i} \left(\frac{1 + R_i}{2}\right)^{\delta_i} \left(\frac{1 - R_i}{2}\right)^{\delta_i'}$$

where $c_i = \frac{1}{2}(x_i + \tilde{x}_i)$, $\delta_i = 1$ if $X_i = (-1, 1)$ and 0 otherwise and $\delta_i' = 1$ if $X_i = (1, -1)$ and 0 otherwise. This gives the following Hamiltonian for the two-state net:

$$H_\tau^{Y,2}(X) = \sum_{i=1}^N \tfrac{1}{2}|(x_i + \tilde{x}_i)| V_i - \tau \sum_{i=1}^N \log z_i(X)$$
$$- \tau \sum_{i:X_i=(-1,1)} \log\left(\frac{1+R_i}{2}\right) - \tau \sum_{i:X_i=(1,-1)} \log\left(\frac{1-R_i}{2}\right).$$

Again we see the level-splitting effect. Observe that this Hamiltonian exists only for $4\exp(-2V_i/\tau) \leq 1$, which is precisely the condition for the Yedidia net to have a two-state representation. Finally let us consider the zero-noise limit of the above Hamiltonians. Define the index sets

$$\Theta = \{i : |h_i| \geq V_i\} \qquad \Theta' = \{i : |h_i| < V_i\}$$

for $h_i = h_i(b)$ or $h_i = h_i(X)$ respectively. For the Little two- and three-state models we then get

$$\lim_{\tau\to 0} H_\tau^{L,2}(X) = \sum_{i=1}^N \tfrac{1}{2}(x_i - \tilde{x}_i) V_i - \sum_\Theta |h_i(X)| - \sum_{\Theta'} V_i$$
$$\lim_{\tau\to 0} H_\tau^{L,3}(b) = - \sum_{i:b_i=0} V_i - \sum_\Theta |h_i(b)| - \sum_{\Theta'} V_i.$$

Now, if $b = \psi(X)$ then $b_i = \frac{1}{2}(x_i + \tilde{x}_i)$, and hence $b_i = 0$ if and only if $x_i = -\tilde{x}_i$, i.e. if and only if $x_i - \tilde{x}_i \neq 0$. Hence the two limits are the same if and only if $x_i \leq \tilde{x}_i$ for all neurons i. In particular the minima of $\lim_{\tau\to 0} H_\tau^{L,2}(X)$ and $\lim_{\tau\to 0} H_\tau^{L,3}(\psi(X))$ are the same. Similarly for the Yedidia model we have

$$\lim_{\tau\to 0} H_\tau^{Y,2}(X) = \sum_{i=1}^N \tfrac{1}{2}|x_i + \tilde{x}_i| V_i - \sum_\Theta (|h_i(X)| - V_i)$$

and

$$\lim_{\tau\to 0} H_\tau^{Y,3}(b) = \sum_{i=1}^N |b_i| V_i - \sum_\Theta (|h_i(X)| - V_i).$$

Hence in this case we in fact have

$$\lim_{\tau\to 0} H_\tau^{Y,2}(X) = \lim_{\tau\to 0} H_\tau^{Y,3}(\psi(X))$$

for all states X. Also, apart from an irrelevant constant, the Yedidia three-state and Little three-state nets have the same Hamiltonians in the $\tau \to 0$ limit:

$$\lim_{\tau\to 0} H_\tau^{L,3}(b) = - \sum_{i:b_i=0} V_i - \sum_\Theta |h_i(b)| - \sum_{\Theta'} V_i$$
$$= - \sum_{i=1}^N V_i + \sum_{i=1}^N |b_i| V_i - \sum_\Theta |h_i(b)| - \sum_{i=1}^N V_i + \sum_\Theta V_i$$
$$= -2 \sum_{i=1}^N V_i + \sum_{i=1}^N |b_i| V_i - \sum_\Theta (|h_i(b)| - V_i)$$
$$= \lim_{\tau\to 0} H_\tau^{Y,3}(b) - 2 \sum_{i=1}^N V_i.$$

As expected, the two-state limiting Hamiltonians agree only on states with $x_i \leq \tilde{x}_i$.

6. Conclusion

We have analysed which three-state networks of the type studied by MHV and Yedidia can be realised as networks of two-state neurons. We have shown that for deterministic nets this can always be done and have derived conditions under which it is possible for noisy nets.

Our results raise several interesting points.

(*a*) MHV study the deterministic properties of their model, and Yedidia studies the zero-temperature limit of his network. Both cases have a two-state representation and we thus conclude that neither MHV nor Yedidia can hope to see any new behaviour as a result of their generalisation from two-state to three-state neurons. This appears to be in direct contrast to their claims of significantly improved performance by three-state nets. We believe that the resolution of this paradox lies in the fact that their three-state formalism is in fact selecting a subset of two-state models with especially good performance. In other words, rather than introducing new behaviour, MHV and Yedidia are restricting attention to a subclass of behaviours with particularly useful features.

(*b*) It is not clear to us whether the significant properties of nets in this subclass are best described using a three-state formalism or a two-state one. In particular, MHV and Yedidia suggest that one of the main advantages of their approach is that the third state of a neuron encodes the lack of relevant information at that neuron. In the two-state representation the equivalent situation is signalled by a disagreement in the states of adjacent neurons. Thus a competitive aspect is brought into the dynamics of the net. As we remarked in section 2, this can be generalised in ways in which the three-state representation cannot be. We thus believe that further work is required to determine which approach is more productive.

(*c*) Finally, as regards performance, we should point out that our two-state representations are replacing one three-state neuron by two-state ones. This is fully in accordance with Yedidia's observation that the storage capacity of his three-state networks is roughly double that of an ordinary two-state network.

(*d*) Given that all deterministic three-state nets do have a two-state representation, we were very surprised to find noisy examples such as the MHV model where no two-state realisation is possible even for arbitrarily low noise levels. Furthermore, the Yedidia and three-state Little models, which have the same deterministic limit as the MHV net, do have two-state representations for sufficiently low noise levels. This illustrates very clearly both the singular nature of the zero-noise limit and the sensitive dependence of the microscopic dynamics on the way in which the noise is introduced.

(*e*) Similarly, the Yedidia and three-state Little models satisfy detailed balance whilst the MHV model does not. This shows just how sensitive the existence of a Hamiltonian description is to the details of the model and underlines the importance of developing more general techniques for analysing noise networks.

(*f*) In common with MHV and Yedidia, we have used parallel dynamics to update neurons. We believe that a similar analysis can be carried out for sequential dyanamics (i.e. where only one randomly chosen neuron is updated at each time step), but it will necessarily be more complicated. This is because if we update one of a pair of two-state neurons representing a single three-state neuron we alter the local field of the neuron in that pair.

(*g*) Most of our analysis of three-state neurons should extend naturally the n-state threshold models. In particular, we expect that all deterministic nets should have two-state representations, whilst more and more stringent conditions will be required for noisy nets to have such realisations. Another class of n-state generalisations of the Hopfield–Little networks are the Potts glass and related models (e.g. Cook 1989, and references therein). Kanter (1987) has investigated the two-state representations of such models. In particular, he shows the equivalence between the Hamiltonian of a multispin system with pair interactions and the Hamiltonian of an Ising-spin system with multispin interactions. Unlike the analysis presented in this paper, (a) Kanter only considers the Hamiltonian context, (b) except when n is a power of 2, his two-state neurons are not independent, and (c) his two-state system uses multispin interactions, rather than the standard pair interactions employed in our work.

(h) Finally, we remark that given the prevalence of binary logic in digital electronic circuitry, any hardware realisation of deterministic three-state networks is likely to be an implementation of their two-state representations. In the case of networks with noise the situation is not so clear, and indeed techniques for constructing noisy nets are still in their infancy. However, at least in the low-noise case, it is still likely that for those nets which do have a two-state representation, a two-state hardware realisation will be easier than a three-state one.

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